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Periodic Solutions of Semilinear Equations of Evolution of Compact Type

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The existence of solutions in a weak sense of $x' + (A + B(t,x))x = f(t,x)$, $x(0) = x(T)$ is established under the conditions that A generates a semigroup of compact type on a Hilbert space H; $B(t,x)$ is a bounded linear operator and $f(t,x)$ a function with values in H; for each square integrable $\varphi(t)$ the problem with $B(t,\varphi(t))$ and $f(t,\varphi(t))$ in place of $B(t,x)$ and $f(t,x)$ has a unique solution; and B and f satisfy certain boundedness and continuity conditions.

INTRODUCTION

A major problem in the extension of techniques applicable to differential equation in $\mathbb{R}^n$ to equations in Hilbert and Banach spaces has been the fact that much stronger continuity hypotheses on the coefficients seem to be needed to ensure the same existence results. Thus it is well known that continuity of coefficients is not sufficient to guarantee existence in a Hilbert space. Pazy [9] has introduced a class of evolution equations such that the evolution operator is compact and for which continuity is sufficient to guarantee existence. Fitzgibbon [10, 11] has extended these results to functional differential equations. In this paper, we will show that the existence theory for boundary value problems of Opial [8] and of Lazar and Leach [5] and many others can be satisfactorily extended, at least in the case of periodic boundary conditions. Ward [13, 14] has discussed boundary value problems for this type of evolution equation. He tackles only the asymptotically sublinear case but for quite general linear and non-linear boundary conditions. We treat here the situation which is asymptotically linear and for which there is uniqueness for solutions of a related equation.

For first order systems of equations in $\mathbb{R}^n$, Opial [8] has established existence of solutions of boundary value problems. He showed that if

$$\text{(i)} \qquad \begin{aligned} x' &= A(t,\varphi(t))x, \\ Bx &= 0 \end{aligned} \tag{0.1}$$

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where B is a bounded linear map from the continuous functions to $\mathbb{R}^n$) has unique solution $x(t) \equiv 0$, for all continuous φ,

(ii) if $b(t, x)$ is continuous in x for almost all t, measurable in t for all x and satisfies

$$\sup_{\|x\| \leqslant k} \|b(t, x)\| = \beta_k(t) \qquad \text{and} \qquad \lim_{k \to \infty} \frac{1}{k} \int_0^T \beta_k(t)\, dt = 0,$$

(iii) if $A(t, x(t))$ lies in a closed bounded (and hence compact) subset of the normed linear space $L_1(M)$ consisting of integrable matrices with norm $\|A\|^\circ = \max_{t \in [0,T]} \sum_{i,j=1}^n \int_0^t a_{ij}(s)\, ds$ for all continuous $x(t)$, then the problem

$$\begin{aligned} x' &= A(t, x)x + b(t, x) \\ Bx &= 0 \end{aligned} \tag{0.2}$$

has a solution. For some further developments of this type, see Kartsatos [12].

Later authors have developed this result. Further results in this direction are those of Lazar and Leach [5]. They show that the boundary value problem

$$\begin{aligned} x'' + h(t, x, x')x &= f(t, x, x'), \\ x(0) = a, \qquad x(T) &= b \end{aligned} \tag{0.3}$$

with h lying between two eigenvalues of the homogeneous problem and f uniformly bounded, has a solution. The conditions imply that the linearized version has a unique solution. For further results see Kannan and Locker [3].

All these paper used fixed point theorems of Schauder type. Kannan and Locker use the spectral theory of self-adjoint operators in Hilbert space and the contraction mapping principle (as outlined in Kazdan and Warner [6]) to obtain the necessary a priori bounds. Estimates on first derivatives seem to be necessary in all the above.

In this paper we will extend this type of result to certain equations of evolution involving unbounded operators. For simplicity, we deal only with periodic boundary conditions. We consider

$$\begin{aligned} x' + (A + B(t, x))x &= f(t, x), \\ x(0) &= x(T), \end{aligned} \tag{0.4}$$

where A is the generator of a semigroup of compact type on a Hilbert space, $B(t, x)$ is a bounded operator and $f(t, x)$ a function. We will prove that the uniqueness of the linearized version for B and f belonging to certain sets implies the existence of a weak solution of (0.4).

Two problems arise in making the above type of extension. Firstly, a priori estimates of the derivative are not easily available in the case of unbounded operators A. These seem necessary in all the above papers. In our case, the compact character of the semigroup ensures the compactness of certain integral operators, and a variant of the principle introduced in Opial [8] enables us to do without estimates of the first derivatives. Secondly, one would like to apply the result to obtain the same type of result as that of Lazar and Leach quoted above, but in infinite dimensional spaces. It would be desirable to formulate a result such as "if λ is not an eigenvalue of $x' + (A + \lambda I)x = 0$, $x(0) = x(T)$ and if for all $\varphi(t) \in L^2$, $\|\lambda I - B(t, \varphi(t))\| < \beta$, where β is the distance from λ to the nearest eigenvalue, then the above nonlinear problem has a periodic solution." The condition $\|\lambda I - B(t, \varphi(t))\| < \beta$ would be shown to imply uniqueness of the linearized problem and hence the quoted result would follow from the general result. But to apply Opial's principle, we will need the fact that those B satisfying $\|\lambda I - B\| < \beta$ lie in a compact set of operators (see condition (iii) above) in some sense of "compact." Such sets are not compact in the strong operator topology, but are compact in the weak operator topology on L^2 as we shall show below (Proposition 1.1). In consequence, we will have to work with weak topologies and weak convergence throughout. The weak formulation thus seems inescapable if we are to have reasonable applications. In Section 2 we will discuss the necessary weak convergence theory for semigroups.

In Section 4 we will give an application of the theory to a first-order equation in which A is self-adjoint. Since the periodic boundary value problem for the equation $x' + Ax = 0$ is not self-adjoint, there may be non-real eigenvalues, so that the situation here is different from the second-order case considered by Lazar and Leach. We will give an estimate for the "distance to the nearest eigenvalue" (β in the above formulation) in terms of an operator norm (see Theorem 4.1).

We have chosen constant A, bounded B and periodic boundary conditions to ensure technical simplicity. Extensions to variable $A(t)$, certain unbounded operators B, other boundary conditions and higher order equations are envisaged. Also, the boundedness of f may be relaxed to asymptotic sublinearity. These results may also be generalized to functional differential equations of retarded type. For a treatment of this aspect see Becker [15].

The paper is divided as follows: Section 1 contains three important preliminary results, one proved in the Appendix and another referred to from Laptev [4]. Section 2 contains convergence results for compact semigroups that are much stronger than those available in the noncompact case. Section 3 has the main theorem and Section 4 has an application to the case where A is self-adjoint.

1. Notation and Preliminary Results

We work in Hilbert space, and all Hilbert spaces will be supposed separable. H, H_1, H', etc. will be used to denote Hilbert spaces.

Given Hilbert spaces H_1 and H_2, $B(H_1, H_2)$ will denote the bounded linear operators from H_1 to H_2, and $B(H)$ will be written instead of $B(H, H)$.

For $I \subseteq \mathbb{R}$ (the reals) a compact interval, we denote by $L^p(I, H)$ $(1 \leqslant p < \infty)$ the space of (equivalence classes of) functions from I to H which are strongly measurable and satisfy $\int_I \|x(t)\|^p \, dt < \infty$. The inner product in H will in these circumstances be denoted by $(\cdot, \cdot)$ and that in $L^2(I, H)$ by $[\cdot, \cdot]$, where $[x(t), y(t)] = \int_I (x(t), y(t)) \, dt$. With this inner product, $L^2(I, H)$ becomes a separable Hilbert space. We write $\|x(t)\|_{L^2} = [x(t), x(t)]^{1/2}$.

Let $\mathscr{L}(I, H)$ denote the set of maps from $I \subseteq \mathbb{R}$ to $B(H, H)$ and $\mathscr{M}(I, H)$ the subset of strongly measurable such maps (i.e., those $B(t)$: $I \to B(H, H)$ such that $B(t)x$ is strongly measurable for each $x \in H$). $\|B(t)\|$ denotes the Hilbert space H-norm of the operator $B(t)$. For details of the measure theory, see Dunford and Schwartz [2, Vol. I].

The following result on weak compactness in $\mathscr{L}(I, H)$ has analogues in the literature, but we have not been able to find the precise version that we need. Since the proof is not long, we include it in the Appendix.

If $B(t) \in \mathscr{M}(I, H)$ and $\|B(t)\| \leqslant M$ $(t \in I)$ then for almost all t, $B(t)$: $H \to H$ and in addition we can regard $B(t)$: $L^2(I, H) \to L^2(I, H)$ as defined by $B(t)(x(t)) = B(t)\, x(t)$ (a.e. for $t \in I$). We denote the latter (bounded linear) transformation by $\tilde{B}$.

Proposition 1.1. *For any $M > 0$, the set*

$$S_M = \{\tilde{B} \mid B(t) \in \mathscr{M}(I, H) \text{ and } \|B(t)\| \leqslant M \text{ a.e. for } t \in I\}$$

is compact in the weak topology in $B(L^2(I, H))$.

Proof. See Appendix.

We state the following result of Laptev [4] as part (a) of the next proposition.

Proposition 1.2. (a) *Let $K(t, s)$: $I \times I \to B(H)$ be compact for almost all (t, s), and let $\int_{I \times I} \|K(t, \tau)\|^2 \, dt \, d\tau < \infty$. Then the map A: $L^2(I, H) \to L^2(I, H)$ defined by $(Ax)(t) = \int_I K(t, \tau)\, x(\tau) \, d\tau$ is compact.*

(b) *Let $K(t, s)$: $I \times I \to B(H)$ be compact for almost all (t, s), and let $\int_I \|K(t, \tau)\|^2 \, d\tau < \infty$ (t fixed). Then the map A': $L^2(I, H) \to H$ defined by $A'x = \int_I K(t, \tau)\, x(\tau) \, d\tau$ is compact for each t for which it is defined.*

Proof (b). I is bounded here, so $\int_{I \times I} \|K(s, \tau)\|^2 \, d\tau \, dt < \infty$ for fixed s and by part (a), the map A defined by the kernel $K(t, \tau)$ with first argument held constant is compact from $L^2(I, H)$ to itself. The image of x under A is constant and so the compactness of A' follows easily.

The following will be needed in the proof of Theorem 3.1. For real-valued functions it is a special case of Krasnoselskii [7, Theorem 2.1]. We give a short proof for completeness.

PROPOSITION 1.3. *Let $f(t, x)$, where $f: I \times H \to H$, be measurable in t for each $x \in H$ and continuous in x for almost all $t \in I$. Further, let $\|f(t, x)\| \leqslant M$. If $x_n(t) \to x(t)$ in $L^2(I, H)$ then*

$$\int_I \|f(t, x_n(t)) - f(t, x(t))\|^2 \, dt \to 0.$$

Proof. By a theorem of Nemytskii (see Krasnoselskii [7, Chap. I, Lemma 2.1]), if $H = \mathbb{R}$ then convergence of $x_n(t)$ to $x(t)$ in measure implies that $f(t, x_m(t))$ converges to $f(t, x(t))$ in measure. The same proof, with $\|\cdot\|$ in place of $|\cdot|$, shows that under the hypothesis of this proposition, if $\|x_n(t) - x(t)\| \to 0$ in measure then $\|f(t, x_n(t)) - f(t, x(t))\| \to 0$ in measure. Given that $x_n(t) \to x(t)$ in $L^2(I, H)$, it follows that $\|x_n(t) - x(t)\| \to 0$ in measure. Hence given $\varepsilon > 0$, we can choose n_0 such that for $n \geqslant n_0$,

$$\operatorname{meas}\left\{t \mid \|f(t, x_n(t)) - f(t, x(t))\| \geqslant \frac{\varepsilon}{2\operatorname{meas}(I)}\right\} \leqslant \frac{\varepsilon}{4M^2}.$$

Then for $n \geqslant n_0$ we have

$$\int_I \|f(t, x_n(t)) - f(t, x(t))\|^2 \, dt$$

$$\leqslant \frac{\varepsilon}{2\operatorname{meas}(I)} \cdot \operatorname{meas}(I) + 2M^2 \cdot \frac{\varepsilon}{4M^2} \leqslant \varepsilon.$$

Hence the result.

2. Equations of Evolution

We consider equations of the form

$$dx/dt + (A + B(t))x = 0, \tag{2.1}$$

where A is an unbounded operator in a Hilbert space H which is closed with dense domain, and $B(t)$ is a bounded linear operator a.e. for $t \in I$. We assume throughout that A is the generator of a strongly continuous semigroup.

Equation (2.1) is said to be of *compact type* if there exists a fundamental solution $\Phi(t, s)$ with the following properties:

(A_1) $\Phi(t, s)$ is a strongly continuous map of $S = \{(t, s) \mid 0 \leqslant s \leqslant t \leqslant T\}$ into $B(H)$, $\|\Phi(t, s)\| \leqslant M$ on S, $\Phi(t, t) = I$ and $\Phi(t, \sigma)\, \Phi(\sigma, s) = \Phi(t, s)$ $(s \leqslant \sigma \leqslant t)$.

(A_2) For any $f \in L^2(I, H)$ and $x_0 \in H$ there is a unique continuous solution $x(t)$, with $(x(t), y)$ absolutely continuous, of

$$\begin{aligned} \frac{d}{dt}(x(t), y) &= -(x(t), (A^* + B^*(t))\, y) + (f(t), y), \\ x(0) &= x_0 \end{aligned} \tag{2.2}$$

for y in the domain of A^*, which is given by

$$x(t) = \Phi(t, s)x_0 + \int_s^t \Phi(t, \tau) f(\tau)\, d\tau. \tag{2.3}$$

(A_3) For $t > s$, $\Phi(t, s)$ is compact, and is continuous in s and t in the uniform norm.

We state a perturbation theorem for equations of compact type.

THEOREM 2.1. *Let $dx/dt + Ax = 0$ be of compact type with fundamental solution $\Phi(t, s)$. Let $B(t) \in \mathscr{M}(I, H)$ satisfy $\|B(t)\| \leqslant M_1$ a.e. for $t \in I$. Then* (2.1) *is of compact type, and its fundamental solution $\Psi(t, s)$ satisfies*

$$\Psi(t, s)x_0 = \Phi(t, s)x_0 - \int_s^t \Phi(t, \tau)\, B(\tau)\, \Psi(\tau, s)x_0\, d\tau \tag{2.4}$$

for $t > s$, $x_0 \in H$. Also

$$\|\Psi(t, s)\| \leqslant K, \tag{2.5}$$

where K depends only on M of (A_1), *and M_1.*

Proof. Standard. See, e.g., Balakrishnan [1, 4.12]. See also Ball [16].

PROPOSITION 2.2. *Let $\Psi(t, s)$ be a fundamental solution of* (2.1) *satisfying* (A_1)–(A_3). *Then if $I_s = [s, T]$*

(a) *The map A: $L^2(I_s, H) \to H$ defined by $x(t) \to \int_s^t \Psi(t, \tau)\, x(\tau)\, d\tau$ is compact for each $s \in I$ and each $t > s$, $t \in I_s$.*

(b) *The map* $B\colon L^2(I_s, H) \to L^2(I_s, H)$ *defined by*

$$x(t) \to \int_s^t \Psi(t, \tau)\, x(\tau)\, d\tau \text{ is compact for each } s \in I.$$

Proof. By (A_3) and Proposition 1.2.

Remark. The usual type of convergence theorem for semigroups says that solutions of $dx/dt + (A + B_n(t))x = 0$ converge strongly if $B_n(t)$ converges strongly. For equations of compact type this may be improved considerably. We prove:

THEOREM 2.2. *Let* $dx/dt + Ax = 0$ *be of compact type, let* $B_n(t) \in \mathscr{M}(I, H)$ *and satisfy* $\|B_n(t)\| \leqslant M_1$ *and let* $\tilde{B}_n(t) \to \tilde{B}(t)$ *weakly in* $B(L^2(I, H))$. *Then if* $\Phi_n(t, s)$ *denotes the fundamental solution of*

$$dx/dt + (A + B_n(t))x = 0 \tag{2.6}$$

we have $\Phi_n(t, s) \to \Psi(t, s)$ *strongly, where* $\Psi(t, s)$ *is the fundamental solution of* (2.4), *boundedly on* $\bar{S} = \{(s, t) \mid s \leqslant t,\ (s, t) \in I \times I\}$ (*i.e.,* $\|\Phi_n\|$ *is majorized on* $\bar{S}$ *by a constant independent of* n).

Proof. By Theorem 2.1, we have

$$\Phi_n(t, s)x_0 = \Phi(t, s)x_0 - \int_s^t \Phi(t, \tau)\, B_n(\tau)\, \Phi_n(\tau, s)x_0\, d\tau \qquad (x_0 \in H). \tag{2.7}$$

Since $\|B_n(t)\, \Phi_n(t, s)x_0\| \leqslant M'$ by (2.5) for fixed s, there exists a subsequence $B_{n_k}(t)\, \Phi_{n_k}(t, s)x_0$ converging weakly in $L^2([s, T], H)$. (For simplity, we will write n instead of n_k in what follows.) But by (A_3) and Proposition 2.2(b) the integral operator in (2.7) is compact from $L^2([s, T], H)$ to itself. Hence from (2.7), $\Phi_n(t, s)x_0$ converges strongly in $L^2([s, T], H)$ with limit $\Psi_1(t, s)x_0$, say.

Now $\tilde{B}_n(t)$ being uniformly bounded and convergent in the weak operator topology on $B(L^2([s, T], H))$ and $\Phi_n(t, s)x_0$ converging strongly implies that $B_n(t)\, \Phi_n(t, s)x_0$ converges weakly in $L^2([s, T], H)$ to $B(t)\, \Psi_1(t, s)x_0$. Hence $\int_s^t \Phi(t, \tau)\, B_n(\tau)\, \Phi_n(\tau, s)x_0\, d\tau$ converges to $\int_s^t \Phi(t, \tau)\, B(\tau)\, \Psi_1(\tau, s)x_0\, d\tau$ strongly in $L^2([s, T], H)$. Taking limits in (2.7) we see that $\Psi_1(t, s)$ satisfies the integral equation (2.4) and by uniqueness (A_2), $\Psi_1(t, s) = \Psi(t, s)$.

Subtracting (2.7) from (2.4) we see that

$$(\Psi(t, s) - \Phi_n(t, s))x_0 = \int_s^t \Phi(t, \tau)\, B_n(\tau)(\Phi_n(\tau, s) - \Psi(\tau, s))x_0\, d\tau$$

$$+ \int_s^t \Phi(t, \tau)(B_n(\tau) - B(\tau))\, \Psi(\tau, s)x_0\, d\tau.$$

For fixed t, and the fact that $\|\Phi(t,\tau)B_n(\tau)\| \leqslant MM_1$, the first integral converges strongly in H to 0. Since $(B_n(t)-B(t))\Psi(t,s)x_0$ converges weakly in $L^2([s,T],H)$ to zero as above, Proposition 2.2(a) implies that, for fixed t, the second integral converges strongly in H to zero. The convergence is bounded on $\bar{S}$ by the boundedness of $\{\Phi_n\}$.

Since every sequence of values of n has a subsequence n_k for which $\Phi_{n_k} \to \Psi$, it follows that the whole sequence converges to Ψ.

COROLLARY 2.3. *Let $h_n(t)$ converge weakly to $h(t)$ in $L^2(I,H)$ and let $B_n(t)$ be as in the statement of the theorem. Then if x_n^0 converges strongly to x^0, the solution $x_n(t)$ of $dx/dt + (A + B_n(t))x = h_n(t)$ satisfying $x_n(0) = x_n^0$ (in the sense of (A_2)) converges strongly and boundedly to $x(t)$, the solution of $dx/dt + (A + B(t))x = h(t)$ satisfying $x(0) = x_0$.*

Proof. Similar to the proof of the theorem.

COROLLARY 2.4. *Under the hypotheses of the theorem, if $\{y_n\}$ is a bounded sequence in H, then for each $t > s$, $\Phi_n(t,s)y_n$ has a strongly convergent subsequence (in H).*

Proof.

$$\Phi_n(t,s)\,y_n = \Phi(t,s)\,y_n + \int_s^t \Phi(t,\tau)\,B_n(\tau)\,\Phi_n(\tau,s)\,y_n\,d\tau. \tag{2.8}$$

Since $\{y_n\}$ and $\{B_n(t)\,\Phi_n(t,s)\,y_n\}$ are uniformly bounded, the compactness of $\Phi(t,s)$ and of the integral operator $\int_s^t \Phi(t,\tau)\cdot d\tau$ (see Proposition 2.2(a)) imply that the righthand side of (2.8) has a subsequence which converges strongly in H.

COROLLARY 2.5. *Under the hypotheses of the theorem, if λ belongs to the resolvent set of $\Phi_n(t,s)$ for all n ($t > s$ fixed) and also to that of $\Psi(t,s)$, then $(\lambda I - \Phi_n(t,s))^{-1}$ converges strongly to $(\lambda I - \Psi(t,s))^{-1}$.*

Proof. Observe that

$$\begin{aligned}&(\lambda I - \Phi_n(t,s))^{-1}x - (\lambda I - \Psi(t,s))^{-1}x\\ &\quad = (\lambda I - \Phi_n(t,s))^{-1}(\Phi_n(t,s) - \Psi(t,s))(\lambda I - \Psi(t,s))^{-1}x,\end{aligned}$$

so, since $\Phi n(t,s) \to \Psi(t,s)$ strongly, it suffices to show that $(\lambda I - \Phi_n(t,s))^{-1}$ is uniformly bounded for $t > s$ fixed. If the latter does not hold, there exists $\{x_n\}$ with $\|x_n\| = 1$ for all n for which the elements $\{y_n\}$ such that $y_n =$

$(\lambda I - \Phi_n(t,s))^{-1}x_n$ satisfy $\lim_{n\to\infty} \|y_n\| = \infty$. Setting $z_n = y_n/\|y_n\|$ and $w_n = x_n/\|y_n\|$ we see that

$$\Phi_n(t,s)z_n = \lambda z_n - w_n \tag{2.9}$$

and $\|z_n\| = 1$, $\lim_{n\to\infty} \|w_n\| = 0$.

By Corollary 2.4 there is a subsequence such that $\{\Phi_{n_k}(t,s)z_{n_k}\}$ converges. Hence by (2.9), $\{z_{n_k}\}$ converges (to z, say) and taking limits in (2.9) we see that

$$\Psi(t,s)z = \lambda z \qquad \text{and} \qquad \|z\| = 1.$$

This contradicts the fact that λ is in the resolvent set of $\Psi(t,s)$.

3. Periodic Solutions

Consider semilinear equations of the form

$$dx/dt + (A + B(t,x))x = f(t,x). \tag{3.1}$$

A *mild solution of* (3.1) on $I = [0,T]$ *satisfying* $x(0) = x_0$ is a continuous function $x(t)$ satisfying

$$x(t) = \Phi(t,0)x_0 - \int_0^t \Phi(t,\tau)(B(\tau,x(\tau))\,x(\tau) - f(\tau,x(\tau)))\,d\tau \qquad (t \in I),$$

where $\Phi(t,s)$ is a fundamental solution of $dx/dt + Ax = 0$.

A *weak solution* of (3.1) on I is a continuous function $x(t)$ for which $(x(t), y)$ is absolutely continuous for y in domain A^* and which satisfies

$$d/dt(x(t),y) = -(x(t),(A^* + B^*(t,x(t)))\,y) + (f(t,x(t)),y) \quad (t \in I)$$

for y in domain A^*.

A *periodic mild* (*weak*) *solution* of (3.1) is a mild (weak) solution satisfying $x(0) = x(T)$.

Remark. It can be shown using the results of Balakrishnan [1, 4.12] or Ball [16] that under the hypotheses on B and f in the following theorem, $x(t)$ is a mild solution of (3.1) satisfying $x(0) = x_0$ if and only if $x(t)$ is a weak solution satisfying $x(0) = x_0$. We will use the "mild" formulation for convenience.

THEOREM 3.1. *Let* $dx/dt + Ax = 0$ *be of compact type. Let* $B(t,x)$

($\in B(H)$) and $f(t, x)$ be measurable in t for each $x \in H$ and continuous in x for $t \in I$. Let $B(t, \phi(t))$ lie in a weakly closed subset $\mathscr{S}$ of $S_{M'}$ for each $\phi(t) \in L^2(I, H)$ (see Proposition 1.1) and let $\|f(t, x)\| \leqslant M''$ ($t \in I$, $x \in H$). Suppose that for all $\tilde{B}(t) \in \mathscr{S}$, the equation

$$dx/dt + (A + B(t))x = 0 \tag{3.2}$$

has unique periodic solution $x(t) \equiv 0$ (in the sense of (A_2)). Then (3.1) has a mild periodic solution.

Proof. We apply the Schauder fixed point theorem. Firstly, we show that if $h(t) \in L^2(I, H)$ and $\|h(t)\| \leqslant M''$ ($t \in I$), then for each B as given in the statement, the equation

$$dx/dt + (A + B(t))x = h(t) \tag{3.3}$$

has a unique periodic solution $y(t)$ (in the sense of (A_2)) and there exists a C depending only on M', M'' and A such that $\|y(t)\| \leqslant C$ ($t \in I$).

Let $\Psi(t, s)$ be the fundamental solution of (3.2) as guaranteed by Theorem 2.1. Then there exists a periodic solution $y(t)$ of (3.3) iff there exists $x_0 \in H$ such that

$$y(t) = \Psi(t, 0)x_0 + \int_0^t \Psi(t, \tau)\, h(\tau)\, d\tau \tag{3.4}$$

and $(I - \Psi(T, 0))x_0 = \int_0^T \Psi(T, \tau)\, h(\tau)\, d\tau$.

By the uniqueness assumption of the theorem, there does not exist x_0 such that $(I - \Psi(T, 0))x_0 = 0$, so by compactness of $\Psi(T, 0)$, $(I - \Psi(T, 0))$ is invertible, i.e., 1 lies in the resolvent set of $\Psi(T, 0)$.

We show that $(I - \Psi(T, 0))^{-1}$ is bounded by a constant depending only on M' and A. If not, there is a sequence $B_n(t)$ satisfying the same conditions as $B(t)$ in the statement such that $(I - \Phi_n(T, 0))^{-1}$ is unbounded, where $\Phi_n(t, s)$ is the fundamental solution corresponding to $B_n(t)$. By Proposition 1.1 there is a $B^0(t)$ satisfying the same conditions as $B(t)$ in the statement and a subsequence n_k such that $\tilde{B}_{n_k}(t) \to \tilde{B}^0(t)$ weakly in $B(L^2(I, H))$. $B^0(t)$ satisfies the uniqueness hypothesis, so $(I - \Psi^0(T, 0))$ is invertible ($\Psi^0(t, s)$ is the fundamental solution corresponding to $B^0(t)$). By Corollary 2.5, $(I - \Phi_{n_k}(T, 0))^{-1}$ is uniformly bounded. The same argument shows that any subsequence contains a bounded subsequence, a contradiction. Hence $(I - \Psi(T, 0))^{-1}$ is uniformly bounded as stated above. Since $x_0 = (I - \Psi(T, 0))^{-1} \int_0^T \Psi(T, \tau)\, h(\tau)\, d\tau$ and since Ψ is bounded by a constant depending on M' and A only (Theorem 2.1) it follows that x_0 is bounded by a constant depending on M', M'' and A only. By (3.4) this also applies to $y(t)$.

Given $\psi(t) \in L^2(I, H)$ define the map G: $L^2(I, H) \to L^2(I, H)$ by $G\psi = \varphi$, where φ is the unique periodic solution (in the sense of (A_2)) of

$$dx/dt + (A + B(t, \psi(t)))x = f(t, \psi(t)). \tag{3.5}$$

Then by condition (A_2) applied to $dx/dt + Ax = 0$,

$$G\psi = \varphi(t) = \Phi(t, 0)x_0 - \int_0^t \Phi(t, \tau)(B(\tau, \psi(\tau))\, \varphi(\tau) - f(\tau, \psi(\tau)))\, d\tau. \tag{3.6}$$

We will show that G is compact and continuous. By the above, $\|\varphi\| \leqslant C$. So if $\{\psi_n\}$ is a bounded sequence in $L^2(I, H)$ then $\{B(t, \psi_n(t))\, \varphi_n(t) - f(t, \psi_n(t))\}$ (where $G\psi_n = \varphi_n$) is uniformly bounded and hence has a weakly convergent subsequence. The integral operator on the right of (3.6) is compact (Proposition 2.2(a)) so there exists a subsequence for which the integrals convergence in $L^2(I, H)$. The initial values $x_0^{(n)}$ corresponding to φ_n are bounded by the above discussion, so there is a weakly convergent subsequence $\{x_0^{(n_k)}\}$. $\Phi(t, 0)$ is compact for $t > 0$, so that $\{\Phi(t, 0)x_0^{(n_k)}\}$ is strongly convergent for each $t > 0$. Since this sequence is uniformly bounded, it also converges in $L^2(I, H)$ using dominated convergence. So $\{G\psi_n\}$ has a subsequence convergent in $L^2(I, H)$ and hence G is compact.

To prove continuity, let $\psi_n \to \psi$ in $L^2(I, H)$. $G\psi_n = \varphi_n$ is given by the analogue of (3.4)

$$\varphi_n(t) = \Phi_n(t, 0)x_0^{(n)} + \int_0^t \Phi_n(t, \tau) f(\tau, \psi_n(\tau))\, d\tau, \tag{3.7}$$

$$x_0^{(n)} = (I - \Phi_n(T, 0))^{-1} \int_0^T \Phi_n(T, \tau) f(\tau, \psi_n(\tau))\, d\tau, \tag{3.8}$$

where Φ_n is the fundamental solution corresponding to $B(t, \psi_n(t))$. By Proposition 1.3 and the properties of $B(t, x)$, it follows that $B(t, \psi_n(t))\, y(t)$ converges for each $y(t) \in L^2(I, H)$ to $B(t, \psi(t))\, y(t)$ in $L^2(I, H)$. So $\tilde{B}(t, \psi_n(t))$ converges strongly to $\tilde{B}(t, \psi(t))$ in $B(L^2(I, H))$. Similarly $f(t, \psi_n(t))$ converges to $f(t, \psi(t))$ in $L^2(I, H)$. Hence by Theorem 2.2, Φ_n is uniformly bounded and convergent in $L^2(I, H)$, and it follows easily that the integrals in (3.7) and (3.8) converge as $n \to \infty$ in $L^2(I, H)$ and H, respectively. Using Corollary 2.5 we can now pass to the limit in (3.8) showing $x_0^{(n)}$ converges to the initial value of $G\psi$, and then in (3.7) showing that $\{\varphi_n\}$ converges to $G\psi$.

We have shown that G is compact and continuous and that its values are uniformly bounded. By Schauder's theorem there is a fixed point $y(t)$ in $L^2(I, H)$. Using (3.6) with both φ and ψ replaced by $y(t)$ we can show continuity of $y(t)$ by standard arguments. So $y(t)$ is the desired mild periodic solution.

Remark. (1) Using Theorem 2.1, it is easily seen that x is a periodic (mild) solution of (3.1) if and only if

$$\frac{d}{dt}(x(t), y) = -(x(t), (A^* + B^*(t, x(t)))\, x(t)\, y) + (f(t), y),$$

$$x(0) = x(T)$$

and $(x(t), y)$ is absolutely continuous, for y in the domain of A^*.

(2) The sort of weakly closed subset of $S_{M'}$ in the theorem that is often useful is a set of the form

$$\{B \in S_{M'} \mid \|\lambda I - B(t)\| \leqslant M''', t \in I\}$$

for some $\lambda \in \mathbb{C}$. This set is the translation of the weakly compact set $S_{M'''}$ and this is weakly compact by the continuity of translation.

(3) Some of the conditions of Theorem 3.1 are stronger than needed. For example, f can be assumed to be sublinear, but the analysis is more complicated.

4. Application with A Self-Adjoint

In this section we consider the nonlinear boundary value problem of Section 3 but with A self-adjoint. We will consider, for convenience, a point $\mu = \frac{1}{2}(\lambda_N + \lambda_{N'})$ midway between consecutive eigenvalues of A. We would expect that if β is less than the distance from μ to the nearest eigenvalue of the periodic boundary value problem for

$$x' + Ax = \lambda x$$

then for all operators $C(t)$ satisfying $\|\mu I + C\| < \beta$, the equation

$$x' + (A + C(t))x = 0$$

has unique periodic solution 0. Since the set of C satisfying $\|\mu I + C\| \leqslant \beta$ is compact in the weak operator topology (See Remark 2, Section 3) we would get existence for the nonlinear problem by Theorem 3.1. We will carry through this programme below, with the modification that instead of β being the actual distance to the nearest eigenvalue, we will only be able to obtain an estimate in terms of a certain operator norm. Note that the situation here is different from that of, e.g., Lazar and Leach [5], since the boundary value problem there was second order and self-adjoint, while the one we deal with is first order and non-self-adjoint.

Throughout this section it will be assumed that.

(C) A is a self-adjoint unbounded operator on the Hilbert space H which is the generator of a strongly continuous semigroup $U(t)$ and that $(\lambda I + A)^{-1}$ is compact for some $\lambda > \omega_0 = \inf_{t>0}(1/t) \log \|U(t)\|$. It is well known (see, e.g., Balakrishnan [1, Corollary 4.4.2]) that $U(t)$ is self-adjoint for $t \geqslant 0$ and compact for $t > 0$. Further, if $\{\lambda_k\}_{k=1}^{\infty}$ is the set of eigenvalues of A with corresponding complete set of eigenfunctions $\{\phi_k\}$, then we have

$$U(t)x = \sum_{k=1}^{\infty} e^{-\lambda_k t}(x, \phi_k)\phi.$$

By (3.4), if $(I - e^{\mu T}U(T))$ is nonsingular, the equation

$$x' + (A - \mu)x = f(t) \tag{4.1}$$

has unique mild periodic solution

$$\begin{aligned} x(t) &= e^{\mu t}U(t)[I - e^{\mu T}U(t)]^{-1} \int_0^T e^{\mu(T-\tau)} U(T-\tau) f(\tau)\, d\tau \\ &\quad + \int_0^t e^{\mu(t-\tau)} U(t-\tau) f(\tau)\, d\tau \qquad (4.2) \\ &= \int_0^T G_\mu(t, \tau) f(\tau)\, d\tau, \end{aligned}$$

where

$$\begin{aligned} G_\mu(t, \tau) &= (I - e^{\mu T}U(T))^{-1} e^{\mu(T+t-\tau)} U(T+t-\tau) + e^{\mu(t-\tau)} U(t-\tau) \qquad (\tau \leqslant t) \\ &= (I - e^{\mu T} U(T))^{-1} e^{\mu(T+t-\tau)} U(T+t-\tau). \qquad (\tau > t) \end{aligned}$$

Define the operator

$$\mathscr{G}_\mu \cdot = \int_0^T G_\mu(t, \tau) \cdot d\tau \text{ mapping } L^2(I, H) \to L^2(I, H).$$

$\mathscr{G}_\mu$ is compact by Proposition 1.2(a).

THEOREM 4.1. *Let A satisfy* (C) *above, and let $B(t, x)$ $(\in B(H))$ and $f(t, x)$ be measurable in t for each $x \in H$ and continuous in x for $t \in I$. Let f be uniformly bounded. Let $B(t, \varphi(t)) \in \mathscr{S}'$ for all $\varphi(t) \in L^2(I, H)$, where*

$$\mathscr{S}' = \{C(t) \in \mathscr{M}(I, H) \mid \|\tfrac{1}{2}(\lambda_N + \lambda_{N'})I + C(t)\| \leqslant \alpha < \|\mathscr{G}_{1/2(\lambda_N + \lambda_{N'})}\|^{-1}$$
$$\textit{for some pair of consecutive eigenvalues } \lambda_N \neq \lambda_{N'} \textit{ of } A\}.$$

Then (3.1) *has a solution.*

Proof. We must show that if $\varphi(t) \in L^2(I, H)$ then $B(t, \varphi(t))$ lies in a bounded weakly closed subset. But $B(t, \varphi(t)) \in \mathscr{M}(I, H)$ and hence lies in $\mathscr{S}'$. Remark 2 after the proof of Theorem 3.1 ensures that $B(t, \varphi(t))$ lies in such a subset. It then remains to show that for $C(t) \in \mathscr{S}'$, the equation

$$x' + (A + C(t))x = 0, \tag{4.3}$$

$$x(0) = x(T), \tag{4.4}$$

has unique mild solution 0. We will set $\mu = \frac{1}{2}(\lambda_N + \lambda_{N'})$ in (4.1) and show that $(I - e^{\mu T} U(T))$ is nonsingular so that (4.2) applies.

$$e^{\mu T} U(T) x_0 = \sum_{1}^{\infty} e^{-(\lambda_k - \mu)T} (x_0, \phi_k) \phi_k$$

and thus $(I - e^{\mu T} U(T))$ can only be singular if there is an $x_0 \neq 0$ such that

$$\sum_{1}^{\infty} (e^{-(\lambda_k - \mu)T} - 1)(x_0, \phi_k)\phi_k = 0,$$

i.e., iff

$$\sum_{1}^{\infty} |e^{-(\lambda_k - \mu)T} - 1|^2 \, |(x_0, \phi_k)|^2 = 0. \tag{4.5}$$

But $e^{-(\lambda_k - \mu)T} \neq 1$ for any k so (4.5) implies that $(x_0, \phi_k) = 0$ (all k), i.e., $x_0 = 0$.

If there is a periodic solution of (4.3) it must satisfy

$$x' + (A - \mu)x = -(C(t) + \mu)x$$

and hence by (4.2)

$$x(t) = -\int_0^T G_\mu(t, \tau)(C(\tau) + \mu)\, x(\tau)\, d\tau.$$

So we have, if $x \neq 0$,

$$\begin{aligned} \|x\|_{L^2} &\leqslant \|\mathscr{G}_\mu\| \, \|(C(t) + \mu)\, x(t)\|_{L^2} \\ &\leqslant \|\mathscr{G}_\mu\| \sup \|C(t) + \mu\| \, \|x\|_{L^2} \\ &\leqslant \|\mathscr{G}_\mu\| \, \alpha \, \|x\|_{L^2} \\ &< 1 \cdot \|x\|_{L^2} \text{ by definition of } \mathscr{S}'. \end{aligned}$$

This is a contradiction, so $x = 0$ as required.

Remark. This theorem can be applied to parabolic equations, with

$$A = \sum_{i=1}^{n} a_i(\mathbf{x})(\partial^2/\partial x_i^2) \text{ on a suitable finite domain.}$$

Many results are available if A is negative definite, but there are not many treatments of the behaviour at positive eigenvalues, as is afforded by the above theorem.

APPENDIX: PROOF OF PROPOSITION 1.1

For simplicity, we will assume measure$(I) < \infty$, although this is unnecessary. We abbreviate $L^2(I, H)$ to L^2 below. Since

$$\|B(t)\,x(t)\|_{L^2} \leqslant M\,\|x(t)\|_{L^2}$$

it follows that for each $x \in L^2$, $\{\tilde{B}x \mid \tilde{B} \in S_M\}$ is weakly compact in L^2. It then follows from the Tychonov product theorem that if S_M is weakly closed, it is weakly compact (see Dunford and Schwartz [2, Vol. I, Chap. VI.9.2]).

To prove closure, we must show that if $\{\tilde{B}_n\} \subseteq S_M$ is weakly convergent, then there exists $B(t) \in \mathscr{M}(I, H)$ such that $\|B(t)\| \leqslant M$ and $\tilde{B}_n \to \tilde{B}$ in the weak operator topology in $B(L^2(I, H))$. Let $\{z_k\}$ be dense in H. Then $B_n(t) z_k$ converges weakly in L^2 to $w_k(t) \in L^2$ (all k). Define $B(t) z_k = w_k(t)$. This defines $B(t)$ a.e. at z_k and the set E_0 of all t such that $B_n(t) z_k$ does not converge for some k has measure zero (being a countable union of sets of measure zero). So $B(t)$ is defined on $\{z_k\}$ except for $t \in E_0$. The set $\{x(t) \in L^2(I, H) \mid \|x(t)\| \leqslant M$ a.e. for $t \in I\}$ is strongly closed in L^2 and convex, and hence weakly closed in L^2. Thus from $\|B_n(t) z_k\| \leqslant M\,\|z_k\|$ a.e. it follows that $\|B(t) z_k\| \leqslant M\,\|z_k\|$ a.e. Let

$$E_1 = E_0 \cup \{t \mid \|B(t) z_k\| > M\,\|z_k\| \text{ for some } k\}.$$

Then as we argued for E_0, E_1 has measure zero. By the denseness of $\{z_n\}$ we may extend the definition of $B(t)$ to the whole of H by continuity for all $t \notin E_1$, and $\|B(t)\| \leqslant M$. $B(t)$ is clearly measurable. We show that $B_n(t) \to B(t)$ in the weak operator topology in L^2.

Given $x, y \in L^2$, $y \neq 0$,

$$\begin{aligned} |[(B_n(t) - B(t))\,x(t), y(t)]| \leqslant{} & |[B_n(t)(x(t) - u_k(t)), y(t)]| \\ & + |[(B_n(t) - B(t)) u_k(t), y(t)]| + |[B(t)(u_k(t) - x(t)), y(t)]| \end{aligned}$$

for any $u_k(t) \in L^2$. So if we find a dense set $\{u_k(t)\} \subseteq L^2$ for which $\tilde{B}_n u_k \to \tilde{B} u_k$ weakly (all k), we will be done. Let $\{I_l\}$ be the class of closed subintervals of I having as endpoints either rationals or endpoints of I. Then the set of all functions of the form

$$\sum_{j=1}^{N} z_{n_j} C_{I_{l_j}}$$

for some N, $\{z_{n_j}\} \subseteq \{z_n\}$, $\{I_{l_j}\} \subseteq \{I_l\}$ (C_{I_l} is the characteristic function of I_l) is countable, dense and since

$$\int_I ((B_n(t) - B(t))\, C_{I_{l_j}} z_{n_j}, y(t))\, dt$$

$$= \int_I ((B_n(t) - B(t)) z_{n_j}, C_{I_{l_j}} y(t))\, dt \to 0$$

this sequence has all the desired properties.

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